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# PROCEEDINGS

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# DISCUSSION OF PROCEEDINGS - SEPARATES

292, 330

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# DISCUSSION OF STRENGTH OF COLUMNS ELASTICALLY RESTRAINED AND ECCENTRICALLY LOADED PROCEEDINGS-SEPARATE NO. 292

ROBERT L. KETTER,* J.M. ASCE and LYNN S. BEEDLE,* A.M. ASCE.—The work of the project at Cornell University sponsored by the Column Research Council and the Bureau of Public Roads, is a real contribution to engineering knowledge. Nearly every structural compression member is a restrained beam-column and yet most current specifications are based on the strength of the pin-ended member although an estimate is made of the effective length due to end restraint. The authors have developed a solution to the elastically restrained column problem by limiting the study to members that do not fail by local or lateral-torsional instability. This solution will be basic to future work in this field.

It is the purpose of this discussion to consider some aspects of the experimental results, to present a precise method for determining the plastic reduction factor, $\eta$, to examine the "shape factor" mentioned by the authors, and to suggest a tentative form for design curves.

With regard to the nomenclature it is understood that the term "$\eta$" is the plastic reduction factor. However, on page 14, $\gamma'$ is called the "plastic reduction factor". This should be cleared up.

The term "c" used by the authors is apparently a factor less than the Euler length factor, K[17]. (See Fig. 5b.) However, in the earlier Cornell work[8] which is used in the paper, there is no question but that "c" is equal to the Euler length factor. Comment on this point seems in order.

The use of a square shape is justified for checking theory. The testing of a limited number of I-shapes bent about the weak axis is also justified to confirm the findings, thus eliminating the variable of lateral-torsional instability.

It is considered that attention should next be directed, however, to the wide-flange shape with flexure forced about the strong axis, since it is expected that most structural columns will be loaded in this manner. Lateral-torsional buckling will occur when columns without complete lateral bracing are bent in this manner. A lower carrying capacity results[18], and theoretical studies are currently under way at Lehigh in an effort to predict this behavior.

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17. "The Basic Column Formula", Column Research Council Technical Memorandum No. 1, May, 1952.
18. "Plastic Deformation of Wide-Flange Beam-Columns", by R. L. Ketter, E. L. Kaminsky and L. S. Beedle, ASCE Proceedings-Separate #330 (Vol. 79), October, 1953.

The use of whitewash as a strain-indicating device is certainly valuable, its use extending back for many years. Lyse and Johnston[19] used this technique. R. S. Johnston[20] used a mixture of portland cement and water for the same purpose. The objective, of course, is to more clearly reveal the flaking of mill scale.

The authors have used a factor of 15% to account for the fact that indication of first yield by whitewash was consistently above the computed value. This was done both for as-delivered and annealed members. This delayed indication of yielding by whitewash might also be attributed to an irregular mill scale or else the flaking might not have been observed when it first developed. A considerable body of test data at Lehigh University indicates that for typical wide-flange shapes yielding occurs at less than (not greater than) the nominal computed value. The cause is attributed to residual stresses. Considering such initial stresses, fair agreement has been obtained between the theoretical yield load and the load at which flaking of whitewash was first observed. (For example see Figs. 27-31 of Ref. 18. See also Ref. 21.)

No issue is taken with the fact that an upper yield point effect exists. But it is doubtful if it can account for any such increase as shown, for example, in Fig. 15(b).

## Influence of Residual Stress

The authors mention "a minor effect of residual stress" as indicated by the rounded knee of the curve in Fig. 13 for the as-delivered I-section. Recent studies have shown that similar deviations from linearity have a significant effect.[21] From Fig. 22 of Ref. 14, the maximum strength of the as-delivered I-shape column tested by the authors at zero eccentricity is 94 kips or an average stress of 34 ksi (area = 2.76 sq. in.). The yield stress level for this as-delivered 4I9.5 material is 38.0 ksi. Thus the reduction in strength is 10% in a region in which a compression member should carry full yield point stress in the absence of residuals and eccentricities. ($cL/r = 51.8$)

It is to be expected that the residuals would be less in the more compact I-shape than in most wide-flange shapes. But it must also be remembered that the stress-strain diagram on which the authors base their theory (Fig. 1) has a shape not unlike the "cross-section" test of the as-delivered I-shape, as presented in Fig. 13. (Chwalla's work was based on a steel whose proportional limit, $\sigma_p$, was 0.80 times the yield

19. "Structural Beams in Torsion", by Inge Lyse and B. G. Johnston, Transactions, ASCE, V. (1935) .
20. "Compressive Strength of Column Web Plates and Wide Web Columns", by R. S. Johnston, U.S. Bur. Standards Technologic Paper 327, V. 20 (1926) pp. 33-82.
21. "Residual Stress and the Compressive Strength of Steel", by A. W. Huber and L. S. Beedle, Fritz Lab. Report No. 220A.9, Lehigh University, Dec. 1. 1953. (Scheduled for publication in Welding Journal).

stress level, $\sigma_y$. In Fig. 13, $\sigma_p = 0.75\ \sigma_y$.) Thus the influence of residuals had already been taken into account, empirically, in the theory.

Of course as the eccentricity ratio $ec/r^2$ increases the influence of residuals diminishes rapidly (as shown in Ref. 18), until, for a member subjected to bending alone, there is no influence upon the load-carrying capacity.

## Plastic Reduction Factor

The authors' solution is based on the premise that for the given full-length column which is eccentrically loaded and elastically restrained there corresponds a shorter, hypothetical, pin-ended column subjected to an end eccentricity of load such that it behaves in the same manner as the original member. The solution, then, is basically that of determining the equivalent, hypothetical member. Knowing this length and the adjusted eccentricity, strengths could be predicted based on pin-ended column collapse solutions.

Aside from the use of the moment-distribution equation,

$$\epsilon = \frac{2\eta EK}{2\eta EK + \beta_o} \quad \text{(8)}$$

the most important aspect of the problem is the determination of $\eta$, the plastic reduction factor. The authors suggest a straight-line approximation, plotted in Fig. 8, which is given as Eq. 25. A method for determining a more precise value of $\eta$ will now be examined—a method that may be applied equally well to any material whose moment-curvature relationship may be obtained.

The plastic reduction factor, $\eta$, first appears in Eq. 1

$$-\eta EI \frac{d^2y}{dx^2} = P \cdot y \quad \text{(1)}$$

Thus, the secant to the moment-curvature curve in the inelastic range is $\eta$ EI. If the M-$\phi$ curve is non-dimensionalized as shown in Fig. 18, then $\eta$ is the secant to this curve. As pointed out by the authors, $\eta$ decreases in value as load increases until the critical load (or moment) is reached. Thus the correct value of $\eta$ is that which exists at the critical load.

A method for obtaining the critical load for an eccentric column was presented in Ref. 18 and it was based on knowledge of the moment-curvature (M-$\phi$) relationship. Methods for obtaining the M-$\phi$ curve including the influence of axial thrust and other variables were also presented there. From the calculations of moment vs centerline deflection leading up to Fig. 38 of Ref. 18 the centerline moment values corresponding to critical load could be obtained. Hence, precise $\eta$ - values may be obtained from Fig. 26 of Ref. 18, for example, for a WF shape bent about the weak axis. The determination of $\eta$ from an M-$\phi$ curve is illustrated in Fig. 18.

These $\eta$-values are plotted in Fig. 19 as influenced by axial load and

slenderness ratio. The curves in Fig. 20 are for a WF shape (8WF31) bent about the strong axis.

Throughout the writers' calculations, it is assumed that the deflected shape is that of a partial cosine curve. This is in contradiction to fact, but, if the boundary conditions are satisfied, an error in shape has but little influence on the critical load.[18] As a matter of fact it appears that the authors make the same simplifying assumption on p. 9.

The advantage of the more precise determination of $\eta$ is that the theory may be applied to any material and, hence, to steel members containing residual stresses. A disadvantage is in the necessity for computing M-$\phi$ curves. These, however, may be non-dimensionalized and once computed for a given shape, further calculation is unnecessary. Further, if design curves are eventually to be used, then the more accurate work seems justified. An additional advantage is that an arbitrary shape factor need not be used. This matter will be considered next.

## Shape Factor

The authors suggest in Table 3 a modification of the shape factors as presented by Bleich on p. 45 of Ref. 9. (It is noted that Bleich's shape factor for "Case 2" is different by 50% from the value suggested by the authors.) In Bleich's method, the "equivalent eccentricity", K, ($K = \varepsilon c/r^2$) is to be multiplied by the shape factor. The authors multiply the effective eccentricity ($\varepsilon$) by the shape factor, $\mu$. Unless $c/r^2$ of the shape under question is the same as $c/r^2$ of the rectangle (this ratio is also equal to A/S) then the two methods are not equivalent. Bleich notes that his values of $\mu$ "must be considered as crude approximations only, since the effect of cross-sectional form upon the buckling strength of eccentrically-loaded columns is by no means cleared up".

One influence of shape may be seen by comparing Fig. 19 and Fig. 20. Another may be seen by comparison of Fig. 37 and Fig. 38 of Ref. 18. and these two curves are re-plotted in part in Fig. 21 herein. An alternate to the more precise determination of $\mu$, however, is not to use it at all. The procedure suggested above for determining the plastic reduction,factor, $\eta$, and the determination of critical loads in eccentric columns outlined in Ref. 18 makes a separate consideration of shape factor unnecessary. It is already inherent in the calculations.

Fig. 22 shows the variation in shape factor for an 8WF31 shape bent about the strong axis. It was derived from Fig. 21. It shows that $\mu$ may not be considered as constant for a given L/r or for a given axial load.

## Comparison of Methods

Referring to the seven steps suggested by the authors as the "Simplified Method" (pp. 24-25), the only significant change of the above procedure is in Step 2 (determination of $\eta$) and Steps 6 and 7. The steps, in outline, are as follows:

(1) (no change) Calculate $\beta_o$.
(2a) Assume effective lengths, cL, for a range of $P/P_y$ values.
(2b) Determine $\eta$ from Fig. 19 for the assumed values of $P/P_y$ and cL/r.
(3) (no change) Check "c" using Ref. 8.
(4) (no change) Repeat Steps 1, 2, and 3 as necessary.
(5) (no change) Using correct $\eta$-values for each $P/P_y$ determine the relation between $\mathcal{E}$ and e from Eq. (8).
(6) Not necessary
(7a) From Fig. 21 obtain $\mathcal{E}/r$ for the various $P/P_y$ - values and cL/r values obtained above.
(7b) Multiply the results of Steps (5) and (7a) to obtain values of $P/P_y$ vs e/r.

The above alternate procedure has been used for the annealed square bars and the results are plotted in Fig. 23 together with the data from the authors' Fig. 16a. This alternate procedure correlates well with experiment and with the authors' procedure.

The authors have suggested that design curves must be developed. The following paragraphs outline one method by which they may be obtained and present some typical curves.

## Design Curves

In the design of a building frame it is expected that a structural unit would be developed based on a preliminary choice of cross-section for each member. The columns in this framework would require checking; that is, with a given resultant eccentricity delivered to the column it is desired to know whether or not the member is stable and, if so, can there be an additional saving in material by decreasing the size of the member.

A sample set of possible design curves is shown in Fig. 24. The following steps would be used in a design problem involving an eccentrically loaded column.

1. Given: A member of known L/r with a known axial load and eccentricity. To Find: Allowable load on the column.
2. Compute $\beta_o$, the spring constant of the elastic restraints. This would be obtained from the analysis of the frame—already completed.
3. For two $\beta_o/EK$ values obtain the allowable load, P, at the given end moment (or eccentricity).
4. Interpolate (or extrapolate) between the values obtained from the two $\beta_o/EK$ curves. Compare the result with the given P.
5. Improve the design by increasing or decreasing the section modulus of the member based on the results of Step 4. Steps 2, 3 and 4 would be followed through, again, as a check.

The striking thing about the use of Fig. 24 is that it is unnecessary to consider the plastic reduction factor, $\eta$, or the Euler length factor "c". These factors appear only in the calculations leading up to Fig. 24. The curves are developed as follows:

1. An effective slenderness ratio, $cL/r$, is selected. (Knowns: EI, $cL/r$)
2. Select $\eta$ from curves like Figs. 19 or 20. (Knowns: EI, $cL/r$, $\eta$, $P/P_y$)
3. From a suitable $M/M_y$ - $\phi/\phi_y$ curve (for example, Fig. 19), determine $M/M_y$ and $\phi/\phi_y$ noting that

$$\eta = \frac{M_{cr}/M_y}{\phi/\phi_y} = \frac{M_{cr}}{\phi}\left(\frac{1}{EI}\right) \qquad (26)$$

4. From $cL/r$ of Step 1 and $P/P_y$ of Step 2, obtain the end moment for the pin-ended eccentrically loaded column, $M_\varepsilon = P \cdot \varepsilon$. This may be done, either from curves like Fig. 21 or by assuming a cosine curve and making use of Step 3 above. (Knowns: EI, $cL/r$, $\eta$, $P/P_y$, $M_\epsilon = P \cdot \varepsilon$)
5. Assume $\beta_o/EK$ and compute $\beta_o/\eta\, EK$.
6. Obtain "c" from Graph II of Ref. 8 and then compute L. (Knowns: EI, $cL/r$, $\eta$, $P/P_y$, $M_\varepsilon = p \cdot \varepsilon$, $\beta_o$)
7. Compute $e/\varepsilon$ from equation 8, as

$$\frac{e}{\epsilon} = 1 + \frac{\beta_o}{2\eta EK} = \frac{M_e}{M_\epsilon} \qquad (8a)$$

   where $M_e$ is the moment at the end of the elastically restrained column of length L.
8. From Step 4 and Step 7 obtain $M_e$ by direct multiplication. This may be expressed as $M_e/M_y$ and there is now known the relationship between the stiffness ($\beta_o/EK$), the full length of member (L), the critical axial load (P) and the end eccentricity (e) or end moment ($M_e$).

One set of curves is needed for each basic shape. The "WF Section" curve shown in Fig. 24 is for the 8WF31 shape. Although there is considerable variation in WF Shapes, it is probable that a few typical $M$-$\phi$ curves covering groups of WF shapes will prove to be sufficiently accurate.

For any given member, of course, the allowable load at a given eccentricity may quickly be determined from Fig. 24. This has been done for the authors' 1-1/2 inch square annealed bars and the results are shown in Fig. 23. A suitable length correction is made due to difference in material properties since Fig. 24 (as well as those preceeding it) was developed for $\sigma_y = 40$ ksi. (The adjusted L/r is found from the expression,

$$(L/r)_a = (L/r)\sqrt{\sigma_y/40.0}\ )$$

The agreement between the results obtained and the tests and the authors' analysis is most satisfactory.

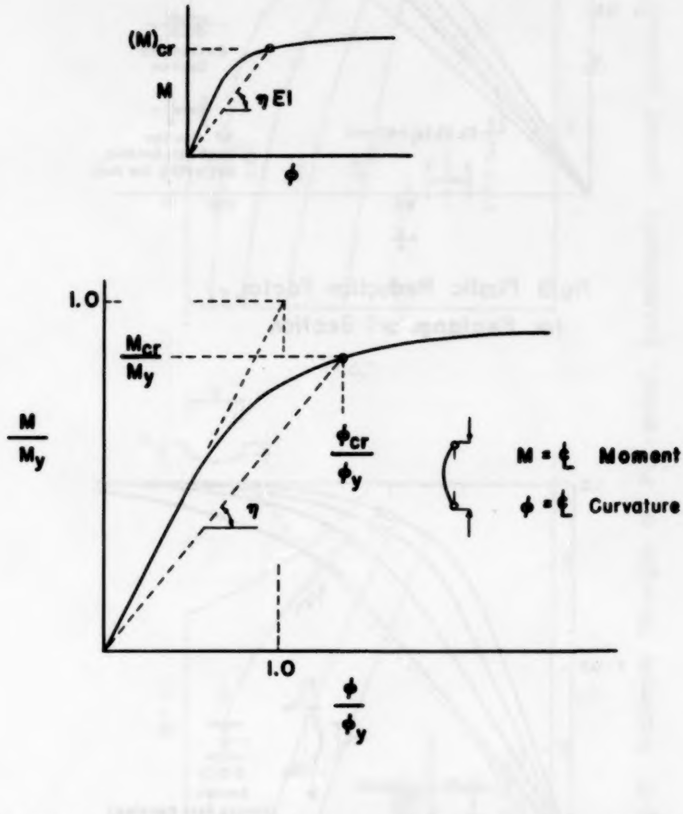


Fig. 18 Typical Moment-Curvature Relationship

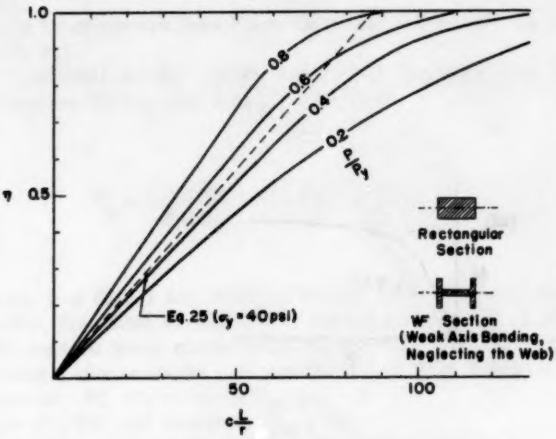


Fig.19 Plastic Reduction Factor, η, for Rectangular Section

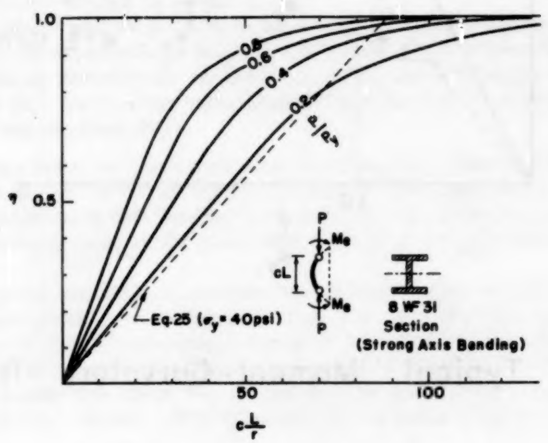


Fig.20 Plastic Reduction Factor, η, for 8 WF 31 Section (Strong Axis Bending)

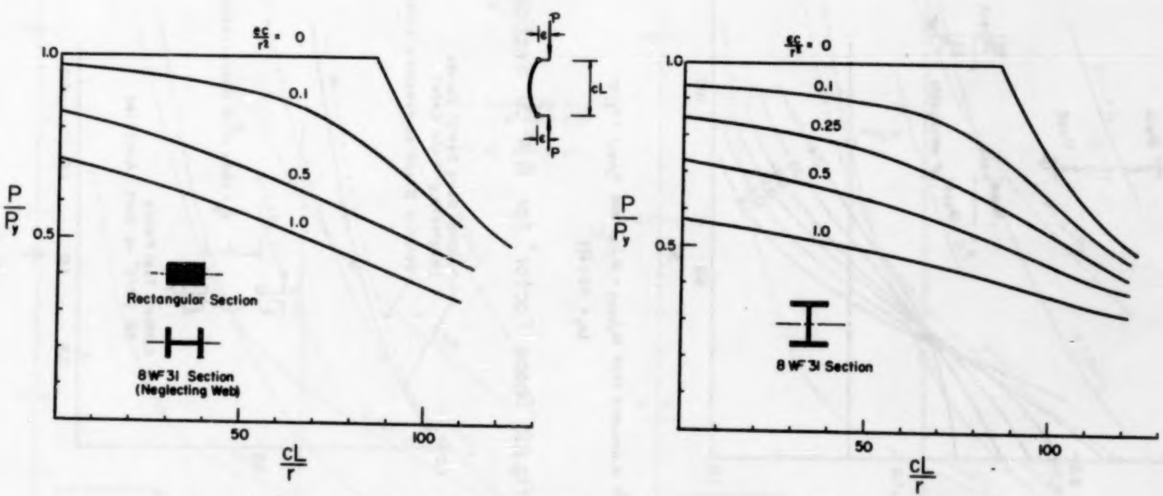


Fig. 21 Maximum Strength of Pin-ended, Eccentrically Loaded Columns[18]

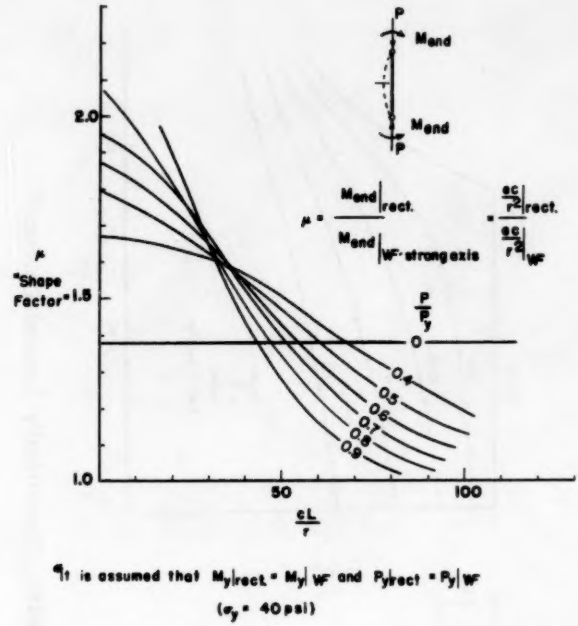


Fig.22 "Shape Factor" for 8 WF 31 Section

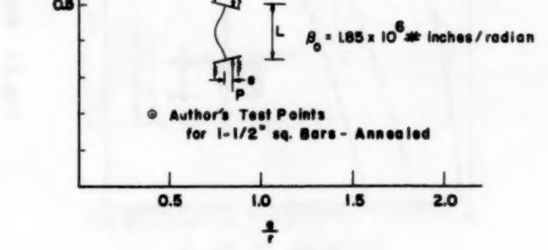


Fig.23 Maximum Strength of Square Bars

$\frac{L}{r} = 80.6$

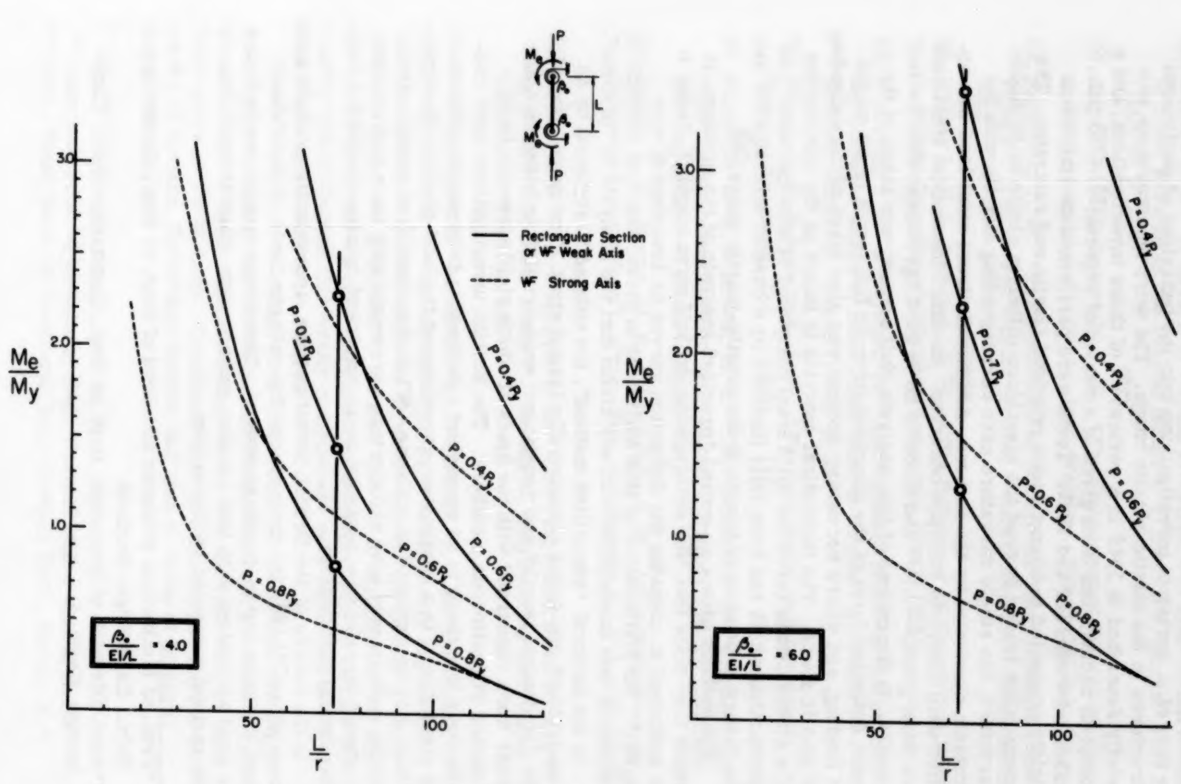


Fig. 24 Design Curves for Elastically Restrained Columns (Based on Maximum Strength)

J. F. BAKER,* A.M. ASCE, and M. R. HORNE.**—The subject discussed by the authors has been studied in Britain for a number of years as part of a series of investigations into the behaviour of continuous structures in the elastic-plastic range. The work on stanchions was briefly described in 1949 in a review(22) of these investigations, and a complete account has been given in a series of reports(23)-(29) published between 1942 and 1949. Tests were performed on stanchions rigidly connected to beams which provided elastic end restraint. The beams were loaded to bend the stanchions either in single or in double curvature, the single curvature case corresponding exactly with the problem discussed by the authors. Additional direct axial load was applied until complete collapse occurred. An exact theoretical treatment has been given,(28),(29) and found to be in good agreement with the test results. In the course of this analysis, full account was taken of the exact deflected form of the stanchion over its full length at all stages of loading, and where necessary account was also taken of the unloading of plastic zones. The theoretical analysis is based on the assumption of a stress strain curve for mild steel composed of straight lines - an assumption which has been fully justified by a careful investigation into the theory of bending of beams in the elastic-plastic range.(30)

The authors give a so-called "precise" analysis of the problem, in which they state that "for determining the half wave length $\ell$ itself, it is sufficient to consider the deflection curves as branches of sine waves". No justification of this statement is given, and it is difficult to reconcile this assumption with any claim that the analysis is "precise".

In the authors' "simplified method", the column is replaced by an "equivalent" pin-ended eccentrically loaded strut. The calculation of the "equivalent length" and "equivalent eccentricity" is based on some other work concerned with the loads at which yield stress is first reached in continuous members. The British investigations show conclusively that there is no consistent relationship between the attainment of yield stress in a continuous stanchion and the load at which that stanchion may be expected to collapse. While the simplified method given by the authors may give satisfactory agreement with their test results, it should therefore be regarded as an empirical treatment with but little theoretical justification.

In the course of the British investigations, approximate methods have been given(24),(26) for determining the collapse loads of stanchions bent in single and in double curvature. These approximate methods are in good agreement with test results, and are very much simpler than the method suggested by the authors.

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P. P. BIJLAARD,* M. ASCE, G. P. FISHER,** A.M. ASCE, and GEORGE WINTER,*** M. ASCE.—The writers sincerely appreciate the valuable contribution made by Messrs. Ketter and Beedle in their discussion and are gratified by the fact that the investigations carried out on different phases of the eccentric column problem at Lehigh University and at Cornell University complement each other and are in basic agreement with each other.

It was a pleasure to receive the discussion of the British investigators, Messrs. Baker and Horne of Cambridge University, and to learn that only part of their important contributions to the title subject had been known to the writers. Comments will be directed first to the remarks of Messrs. Ketter and Beedle, and then to those of Messrs. Baker and Horne.

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Clarifications

A few misunderstandings of terminology may be quickly eliminated:

The discussers are correct in objecting to the designation "plastic reduction factor" for the quantity $\gamma$ on p. 14. It happens that the word "plastic" is here a misprint for "pertinent". This quantity $\gamma$ is correctly designated as "division factor for eccentricity" on p. 18 and should have been so designated wherever else it appears in the paper, to avoid its being confused with the plastic reduction factor $\eta$.

Regarding term "c", this is indeed the Euler length factor as ordinarily defined, provided the concentric buckling stress is in the elastic domain, in which case "c" is a function of $\beta$/EK. It is so used in the earlier Cornell work(8) which presents only elastic theory. However, if the concentric buckling stress is in the plastic domain then this length factor becomes a function of $\beta/\eta$ EK and is smaller than it would be if the column were elastic. The definition of "c" is the same in both cases and the same methods for determining it hold true provided $\beta/\eta$ EK is used instead of $\beta$/EK. This is clearly indicated by Step 3 on p. 24. However, Fig. 5b, to which the discussers refer, concerns an eccentrically loaded column. The effective length cL of the corresponding concentrically loaded column is smaller than the half wave length $\ell$ of the eccentrically loaded column.

In discussing the shape factor Messrs. Ketter and Beedle maintain that the writers apply this factor only to the eccentricity $\varepsilon$ whereas Bleich applied it to the eccentricity ratio $\varepsilon c/r^2$, implying that the writers' method is less general. Apparently they have overlooked Step 7 on p. 25. While in Step 6 the eccentricity $\varepsilon$ is indeed multiplied by the shape factor $\mu$ to obtain $\varepsilon_c$, in the final Step 7 $\varepsilon_c$ appears exclusively in the expression $\varepsilon_c/k_c$. Since the core radius, $k_c = r^2/c$, the shape factor is seen to have been applied to the eccentricity ratio in the same manner as was done by Bleich, as is seen from the following, obvious identities:

$$\frac{\varepsilon_c}{k_c} = (\mu\varepsilon)\frac{c}{r^2} = \mu\left(\frac{\varepsilon c}{r^2}\right)$$

It should be noted here that, for unsymmetrical sections, $k_c$ has to be determined by using the distance "c" from the center of gravity to the concave edge of the cross section, that is the edge where the compressive stress is highest.

Also it may be useful to state that Eq. (16) for $\sigma_{cE}$ applies in the plastic range only, that is, if cL/r is smaller than $(L/r)_E$ from Eq. (17). This is shown clearly in Fig. 11. If cL/r is larger, of course, Euler's formula has to be used, as is indicated on that figure.

As a final point of clarification it should be stated that for all test columns of Tables 4 and 5 and Figs. 15 to 17 the rotational end restraint at both ends was 1850 kip-in/rad. This information was omitted in the paper through an oversight.

Yielding of Steel

It was noted in the paper that indication of yielding by flaking of whitewash seemed to be retarded by about 15% on the average. The discussers

assert that, at least for wide-flange shapes, yielding occurs at loads less than the nominal computed values, that this is attributed to residual stresses, and that with this in mind whitewash flaking was found by them to be an accurate criterion.

The question of whitewash indication is one of experimental technique and will be dealt with at the end of this section. The question of incipient yielding in columns, however, is of primary interest, for instance in connection with fatigue, even though such initial yielding often does not imply failure under static load.

In order to explore this problem, additional tests have been carried out with more accurate instrumentation. These were not completed in time for inclusion in the original paper. Unannealed square bars and I-sections of the same sectional properties were used as given in Table 5. However, end restraints of 985 kip-in/rad. were employed (as compared to 1850 kip-in/rad. in the original series) and the specimens were more slender, having the following lengths:

| | 1-1/2 in. sq. bars | 4 I 9.5 |
|---|---|---|
| Clear or flexible length | 36.00" | 56.00" |
| Distances out-to-out of end blocks | 38.90" | 58.90" |
| Distance between knife-edges | 43.90" | 63.90" |
| Effective slenderness cL/r | 63.7 | 81.0 |

Eccentricities were varied from $e/r = 0$ to $e/r = 2.0$ as in the original tests. Four Tuckerman optical strain gages were mounted at the four corners at mid-length and strain readings were obtained up to loads considerably beyond first yielding.

It is not possible to detect initial yielding by plotting observed strain against load because, in eccentric columns, this relation is non-linear. It is possible, however, to linearize this relationship in the elastic domain in the following manner: For each test load the maximum compression stress at mid-length is computed from the observed mid-deflection and the observed end rotations, on an elastic basis. If this "semi-empirical" stress is plotted against observed strain, the relation must be linear as long as all deformations are elastic, and must become non-linear when yielding starts. Fig. 25 shows four sample curves so obtained, two for the I-section and two for the square bar columns. It is seen that below the yield point the relation is indeed linear and exactly as computed. It is further seen that strain measurements indicate yielding by deviation from the straight line at stresses almost exactly identical with the yield point as determined from sectional compression tests. In contrast, the whitewash flaking is seen always to occur with considerable delay.

Fig. 26 gives the complete test results of these two additional series. It is seen that the ultimate loads of these more slender columns with lesser end restraint are in just as good agreement with the proposed theory as the earlier tests. It is also seen that yielding, as detected by accurate instrumentation, occurs at loads equal to or almost imperceptibly

below the value computed by elastic theory on the basis of the compressive yield point of the material. This is so in spite of the fact that unannealed material was used and that residual stresses were obviously present at least in the I-sections. Though this finding is contrary to the situation as stated by the discussers, it is quite possible that the distribution of residual stresses in these I-sections is sufficiently different from those of the WF-sections tested at Lehigh to make both statements true.

This substantiates the writers' opinion that the flaking of the whitewash does not indicate incipient yielding but is caused by the subsequent formation of macroscopic yield lines at loads higher than the initial yield load. This is also indicated by the fact that the flaking occurs discontinuously along typical yield lines of the type observed in polished specimens. References 12 and 13 show that in relatively thick sections, as used in the tests, the incipient state of stress occurring in these flow lines is not one of pure compression or tension, as in the elastic column, but one of plane strain. Therefore the axial compressive or tensile stress required to initiate a yield layer is $2/\sqrt{3}$ times the lower yield point or 15% above the lower yield point. This, in other words, determines the upper yield point. Although there are other circumstances (speed of testing, nucleation) which may influence the magnitude of the upper yield point, the fact that the whitewash flaking occurred at loads about 15% above the lower yield stress, is in good accordance with this theory. The plastic deformation measured prior to the formation of yield lines may be attributed to localized flow in small, disconnected regions.

## Influence of Residual Stress

Writers and discussers are in agreement that some residual stresses were present in the I-sections. They are also in agreement that Fig. 11, on which the theory is based, implicitly accounts for the presence of some residual stress. They further agree that this influence on column strength is significant only for completely, or nearly, concentric loading, which case is not the subject of the present investigation. The writers wish to add that they doubt whether this relatively small effect warrants, practically, the rather elaborate methods of Reference 18, particularly in view of the erratic character of residual stresses, as to magnitude as well as distribution. This erratic character is illustrated by the observations of the preceding section.

It is not advocated here that the influence of residual stress be discounted which would be identical with assuming the Euler hyperbola to be valid up to the yield point. It seems sufficient, however, to account for this influence, which may be regarded as just one of many "imperfections", by assuming some such transition curve for low L/r as was given on Fig. 11.

## Plastic Reduction Factor

The new concept of the plastic reduction factor was introduced by the writers in order to enable them to reduce the problem of the restrained

column to that of the hinged column. Having introduced this concept they have tried to develop the simplest possible approximate expression for the factor $\eta$, Eq. 25, valid for steels of any yield strength. The prime aim, therefore, was simplicity combined with accuracy, rather than rigor.

It is interesting to see that Messrs. Ketter and Beedle are able to determine a plastic reduction factor by means of their ingenious methods of Reference 18. Whether this method gives "more precise values of $\eta$", as is claimed, is questionable.

While it is quite true that in Eq. 1 the quantity $\eta$ EI is the secant to the moment-curvature curve at one section, it has been pointed out on p. 5 that $\eta$ varies with length "x". Further, the writers' reduction factor $\eta$ serves a double purpose. In the first place it is used for determining the so-called Euler factor "c". But it is also used for calculating that part of the end moment Pe which is distributed to the column. For each purpose a different value of $\eta$ would be needed. However, for practical applications one can find a single value $\eta$ that, used for both purposes, leads to the correct buckling loads. To find this value $\eta$ the discussers propose to determine it only at mid-length and assume a partial cosine curve for the column shape. This is, of course, a crude approximation. In contrast to the discussers' impression, the writers have not made the assumption of a cosine curve (see also the following discussion of Messrs. Baker and Horne's remarks). The expressions for $\eta$ as given by Eq. 25 were obtained by adaptation to Chwalla's "rigorous" though limited theory, to additional data obtained from the more general analytical method outlined on pages 9 and 10 of their paper, and to their own tests. Hence, in all sources on which Eq. 25 is based the eccentric column (a) was elastically restrained and (b) was allowed to assume its natural shape rather than an imposed cosine curve. The authors' value of $\eta$, therefore, represents the integrated effect of a quantity which is actually variable along the column. It could have been called the "effective plastic reduction factor" except for the awkwardness of such a designation. In contrast, the discussers' determination is based on the behavior of only a single section of an eccentric column which is not elastically restrained but is simply supported and whose shape has been approximated by a cosine curve.

It is not implied that the discussers determination is necessarily inferior. However, it can not be regarded as inherently more precise and it is certainly much more complex and laborious. The adequacy of the writers' simple way of determining $\eta$ seems to be proved by the rather extensive test confirmation as given both in the original paper and by the above described additional tests. From these tests it follows that with the writers' value $\eta$, as a function of cL/r only, accurate agreement between theoretical and experimental results is obtained for all ratios $P/P_y$. This shows that, if the writers in their method would use a value of $\eta$ which varies with $P/P_y$, as the discussers do in their Figs. 19 and 20, they would obtain discrepancies between analysis and test.

Shape Factor

As stated in the beginning of this closure, the shape factor $\mu$ is applied to the ratio $\varepsilon/k_c$. Both Bleich's and the writers' shape factor are based on Chwalla's paper (reference 11), where the correction factor $\varphi$ is applied to the core radius $k_c$, so that $\mu\varepsilon/k_c = \varepsilon/(\varphi k_c)$ and $\mu = 1/\varphi$. It should be noted that for unsymmetrical sections $k_c$ should refer to the stress at the concave side of the column. For Case 2 Chwalla's factor $\varphi$, required to obtain the exact eccentric buckling stress for these sections, varies between 0.667 and 1.158, so that $\mu$ varies between 1.5 and 0.87. Therefore the writers used the relatively safe value 1.25, whereas Bleich's value 0.80 is obviously quite unsafe, perhaps caused by an error in interchanging $\mu$ with $\varphi$. The writers are not able to follow the discussers' argument in defining a shape factor. In particular they do not see the significance of the restriction that in order to reduce a WF to a rectangular section by means of a shape factor the yield values $P_y$ and $M_y$ of the two sections have to be the same. This restriction seems foreign to the concept of a shape factor. Apparently owing to their misunderstanding of the way the writer's value $\mu$ was calculated, the shape factors $\mu$ computed by the discussers from their Figs. 21 and given in their Fig. 22, are not correct. For example, from Figs. 21 for $cL/r = 20$ and $P/P_y = 0.9$ the required ratio $ec/r^2 = e/k_c$ for the rectangular and WF sections, the latter bent about their strong axis (Case 4), is 0.19 and 0.135, respectively. Hence, using the computations for rectangular sections, correct results are obtained for the WF section if a ratio $ec/r^2 = 0.19 = 1.41$ (0.135) is used. Therefore the shape factor for $cL/r = 20$ and $P/P_y = 0.9$, from the discussers' own Figs. 21, is $\mu = 1.41$, which is close to the overall value of 1.4 adopted by the writers. However, from the discussers' Fig. 22 they indicate for this same case $\mu = 1.85$. This shows that the great spread in $\mu$ values calculated by the discussers is based on some misconception in their method of calculating $\mu$.

Actually from Chwalla's exact calculation for this case 4, for $P/P_y = 0.37$ the exact value of $\mu = 1/\varphi$ decreases from 1.41 for $cL/R = 33.9$ to a minimum of 1.33 for $cL/r = 75$ and 85 and then increases again to 1.59 for $cL/r = 124.6$. However, from the discussers' Fig. 22, for approximately the same ratio $P/P_y$, namely 0.40, $\mu$ monotonically decreases from 1.67 for $cL/r = 0$ to 1.20 for $cL/r = 100$. It follows from Chwalla's table k, by using $\varphi = 0.7$ or the writers' value $\mu = 1.4$, that the maximum discrepancy in the eccentric buckling stress for this case is only 3%, even though the error in the shape factor $\mu$ is 14%. This illustrates that an excessive refinement in determining shape factors does not seem to be justified.

Comparison of Methods and Design Curves

The modification of the writers' Simplified Method proposed by Messrs. Ketter and Beedle retains all essential characteristics and steps, but introduces the discussers' method of determining the plastic reduction factor "$\eta$" and of accounting for the influence of cross-sectional shape.

Again, the writers do not wish to pass judgement as to the merits of their original method as compared to its proposed modification. It was the writers objective to present as compact and simple a method as possible, valid for all customary shapes, even at the expense of some approximation. For this reason all determinations have been reduced to simple, algebraic formulas.

In contrast, the discussers' modification calls for the construction of such subsidiary charts as their Figs. 19, 20, and 21. Such charts would have to be computed and drawn for a considerable variety of shapes and directions of eccentricity. They would differ for materials with different yield point. Hence, the discusser's method becomes much more complex and laborious than the use of the writers' simple equations.

Writers and discussers are agreed that all proposed methods are too complex for direct practical use and that design curves will have to be derived from them. For that reason the relative degree of complexity of the two variants of the same method is of somewhat secondary importance since it will affect only the amount of work to be expended once for computing the design charts. Such charts are now (1954) being computed at Cornell University by Mr. A. E. Shabaan as part of the originally planned research program.

Messrs. Ketter and Beedle indicate as a special merit of their sample charts that they obviate the necessity of determining the factors "$\eta$" and "c". This, of course, is the very purpose of the design charts and there is, therefore, nothing striking about it. It is only the complexity of determining "$\eta$" and "c" which makes design charts necessary in the first place, and it is the whole point of such charts that "$\eta$" and "c" are needed only in their original construction, but not in their subsequent use.

A sample of the charts developed at Cornell is given on Fig. 27 for a 33,000 psi. yield point and an eccentricity-ratio $ec/r^2 = e/k_c = 3.0$. One practically significant difference between charts of this type and the discussers' Fig. 24 is that in the latter separate curves are needed for the various cross-sectional shapes and the various directions of bending, requiring a very elaborate set of design charts to cover all possible combinations. The writers' charts, in contrast, are constructed in terms of $\mu e/k_c$ and, therefore, are valid for any shape and any direction of bending since these influences are expressed by means of the shape factor, $\mu$.

The chart also illustrates the writers' Conclusion (7), p. 36, to the effect that a given amount of eccentricity has an incomparably smaller effect on a restrained than on a hinged column. Thus, for $L/r = 80$ and the given eccentricity, a column with a restraint factor as small as $\beta/EK = 2.0$ is seen to have a buckling stress of about 16,500 psi, almost twice the buckling stress of about 8,700 psi. which the same column would have if it were hinged.

## British Methods

Messrs. Baker and Horne refer to an exact method and also to a design method, both published by them previously. The discussers'

exact method was not known to the writers but has since been examined in their reference 31. Theirs is a trial and error method, which has to be applied numerically to each case separately. Although very ingenious, this method is extremely laborious and thus has the same drawback as Chwalla's exact method in that it represents exclusively a research method. In contrast it was the writers' aim to develop satisfactorily accurate methods which are sufficiently simple so that they can be used in practical design, either directly or by means of simple charts. Also, for columns in double curvature (equal restraints and equal and opposite eccentricities) discussers' method does not lead to correct results, since they assume that up to and through failure the center of the column does not deflect. Hence they actually consider a column of length L/2 that is elastically restrained and eccentrically loaded at one end and simply supported and concentrically loaded at the other. In fact, however, when the buckling load is reached the center of the column deflects. This well-known fact is confirmed by the discussers' own tests. As explained in reference 3 at the moment of buckling a bifurcation occurs, and the column deflects in a superimposed symmetrical deflection. This happens after the half wave length of the deflection curve becomes equal to the half wave length in which the column would buckle symmetrically, so that with elastic restraints this half wave will be longer than half the length of the column. This means that the buckling stress is less than that found from the discussers' calculation.

According to the discussers their approximate methods (their references 26 and 28) are in good agreement with test results. This may be true for their tests on columns with small L/r ratios and large end restraints (large $\beta$ ), but these methods certainly cannot be used over the entire range of slendernesses and end restraints. For example, for the case of the writers' Fig. 4a from discussers' references 26 and 29 the critical load is given by the formula

$$P_{cr} = \pi \frac{M_p^s}{\theta L}$$

where $M_p^S$ is the ultimate bending moment that can be resisted by column AB and $\theta$ is the angle rotation of the restraining members CA and BD at A and B, caused by moments $Pe + M_p^S$ at A and B, respectively. If the restraint is small, say $\beta = 0$, then $\theta$ will be infinite, while $M_p^S$ is finite, so that $P_{cr}$ would erroneously be found to be zero. This shows that the formulas proposed by the discussers are not applicable for small values of the rotational restraints.

Further, the discussers' methods do not apply for L/r values larger than about 50, since for zero eccentricity the above equation yields a buckling stress equal to the yield stress for all slenderness ratios. Evidently for concentric load and large L/r a correct method has to result in the Euler stress, rather than the yield stress.

A study of Messrs. Roderick and Baker's approximate method (reference 26 and 28) had been made by the writers before they started on their own investigation. The above conclusions were stated somewhat

more elaborately in reference 3, from which the following sentences are quoted (reference 3, page 93):

"Unfortunately this study (of Baker's and Roderick's method) proved that their approach gives reasonably reliable results only for a very limited range of variables (slenderness, restraints, eccentricity) and cannot be generalized to cover the entire field. It was for this reason that the present elaborate study was found necessary."

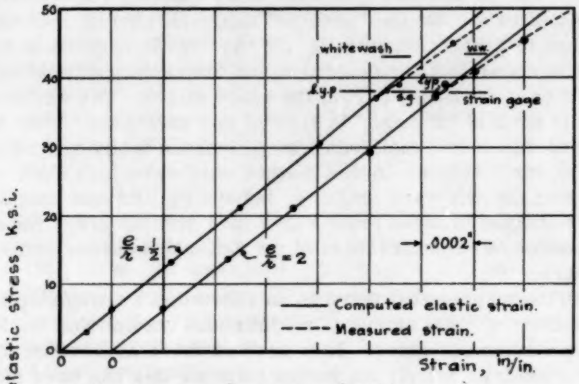
The justification for the "precise" analysis, shortly mentioned by the writers in connection with Eq. (3) and Fig. 7, is given in reference 3, where it is shown that for several cases the resulting buckling stresses do not differ more than 3% from the exact values. The writers have called this method "precise" in view of this precision. They have not maintained that it is a mathematically rigorous method, such as Chwalla's, for instance. In this method sine waves are used only for determining the half wave length $\ell$, but the equilibrium load for these half wave lengths is found from Fig. 7 or a similar graph for other materials, based on the exact form of the deflection curves rather than on a sine wave.

The writers agree that there is no consistent relationship between the attainment of yield stress in a continuous column and the load at which the column collapses. They have nowhere maintained that there is. For eccentric, simply supported columns this has been shown by H. M. Westergaard and W. R. Osgood as early as 1928. It was one of the main objects of this paper to confirm this very fact for end-restrained columns. What the writers have done in their simplified method is this: They have taken Miss Lee's method which applies only in the elastic range, that is, up to the yield point, and have generalized it by means of the plastic reduction factor $\eta$. By this means it was possible to obtain a relatively simple method for analyzing columns in the post-yield range.

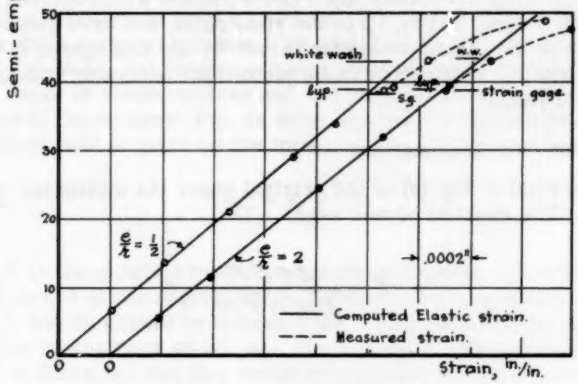
Corrections for Transactions

At the right of Eq. (8) of the original paper the multiplier $\underline{e}$ was omitted. The equation should read:

$$\mathcal{E} = \frac{2\eta EK}{2\eta EK + \beta} e$$




(a) UNANNEALED I SECTION 4 I 9.5




(b) UNANNEALED SQUARE BARS $1\frac{1}{2}$"□

FIG. 25. SEMI-EMPIRICAL STRESS-STRAIN CURVES FOR EXTREME COMPRESSION FIBER AT MID-HEIGHT.

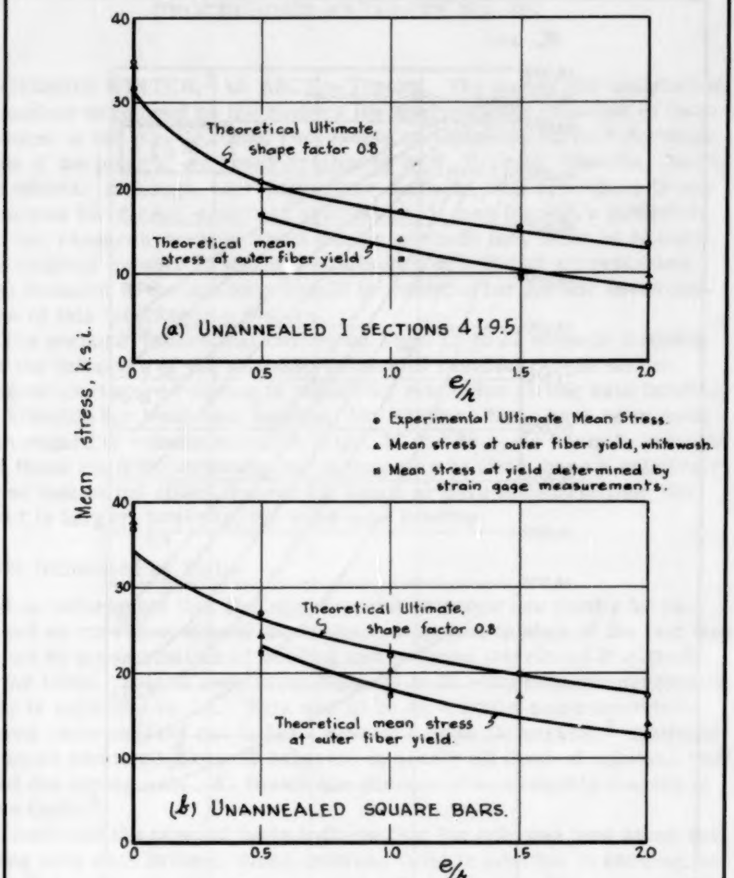



FIG. 26. TEST RESULTS, $\beta_o$ = 985 kip-in./rad.

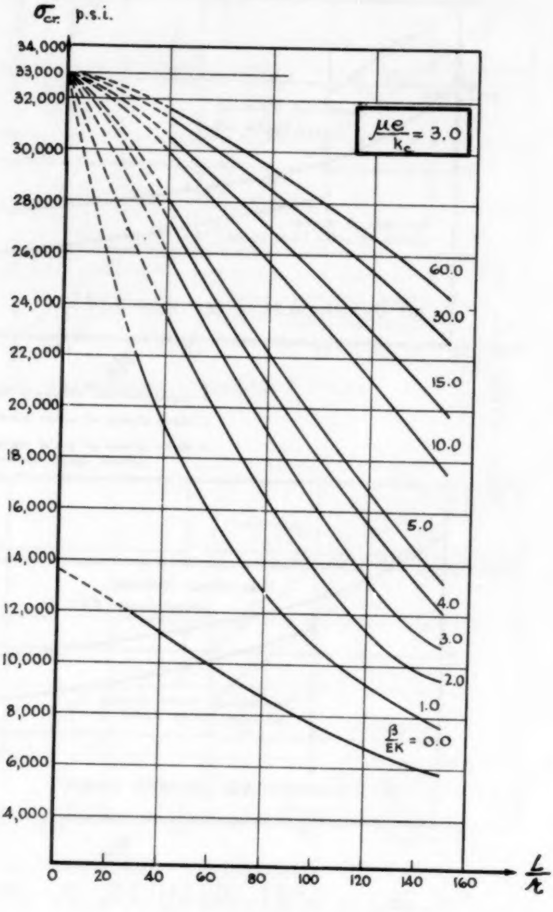



FIG. 27 SAMPLE DESIGN CHART FOR ECCENTRIC BUCKLING STRESS.

## DISCUSSION OF PLASTIC DEFORMATION OF WIDE-FLANGE BEAM-COLUMNS PROCEEDINGS-SEPARATE NO. 330

GEORGE WINTER,[1] M. ASCE.—Theory. The numerical integration procedure developed by the authors for analyzing the behavior of beam-columns in the plastic range represents an ingenious further development of the general approach originated by v. Karman, Chwalla, Jezek, and others. Although, like these other methods, this procedure is too laborious for direct, practical application, it may furnish a powerful tool for research work on which design methods may later be based. The original manner in which the influence of residual stresses has been included in the analysis should be valuable for further investigations of this troublesome feature.

The authors' theoretical curves on Figs. 25 to 31 seem to indicate that the influence of the assumed pattern of residual stress on the moment-curvature relation is relatively minor for strong axis bending, but sizeable for weak axis bending, for which no tests have been made. With regard to column strength, Figs. 37 and 38 again seem to indicate that these residual stresses, for strong axis bending, have a relatively minor weakening effect, except for small or zero eccentricities; the effect is larger, however, for weak axis bending.

### Tests Influenced by Twist

It is unfortunate that the experimental evidence can hardly be regarded as corroboration of the analytical method in view of the fact that failure by a combination of bending and twisting interfered in almost all the tests. This is seen from Figs. 27 to 31 where the occurrence of twist is indicated by LA. This was to be anticipated since eccentric column tests carried out in 1942, also at Lehigh University,[2] exhibited the same torsional-flexural behavior in nearly all tests of columns bent about the strong axis. F. Bleich has already pointed to this feature of those tests.[3]

These and the present tests indicate that for columns bent about the strong axis such failure, which involves twist in addition to bending, is of considerable practical consequence. To the writer this evidence of the importance of twist is one of the chief values of the authors' tests. Thus, it is somewhat disquieting to learn, from the authors' Figs. 35 and 40, that the weakening influence of twist is sizeable even for

1. Prof. and Head, Dept. of Structural Eng., Cornell Univ., Ithaca, N. Y.
2. B. G. Johnston & L. Cheney: Steel Columns of Rolled Wide Flange Section, Am. Inst. of Steel Constr., Progress Report 2, 1942, New York.
3. F. Bleich: Buckling Strength of Metal Structures, p. 47, McGraw-Hill, New York, 1952.

slenderness ratios as low as $L/r = 41$, amounting in this case to a reduction of moment capacity of about 15%. The quoted test results seem to indicate that this feature may prove to be of far greater consequence in reducing the strength of such members than the influence of residual stresses and that further investigations of flexural-torsional failure of steel columns in the plastic range are urgently called for.

Residual Stresses may have Different Patterns

With regard to residual stresses the authors base their analytical findings on an assumed residual stress pattern, Fig. 7, which follows closely the measured stress distribution of Fig. 6, obtained on an 8 WF 31 section. All computed curves, such as Figs. 25, 26, 37, and 38, specifically refer only to this particular section and the authors do not claim general validity for other shapes. A warning may still be in order to the effect that the residual stress pattern of Figs. 6 and 7, while undoubtedly valid for the investigated section, may not be typical for rolled I-shaped sections in general. The authors' pattern results in an average residual compression stress in the flanges, and an average residual tension stress in the web, the former being essentially the cause for the weakening influence of these stresses. There is some evidence that average residual cooling stresses of exactly the opposite distribution have been observed. Average residual tension, rather than compression, stresses in flanges of 9 JB 7.5 and 7 JB 5.5 Junior Beams ranging from 2500 to 19,000 psi. have been reported recently.[4] In German wide-flange sections, 42.5 cm. deep (about 17 in.) residual average compression, rather than tension, stresses of about 24,000 psi. have been measured in the web.[5] It would seem that the pattern of the cooling stresses depends very largely on the details of the distribution of material in flanges and web which, of course, affect the nature of the cooling process. It is possible that the authors' results, obtained on a section with depth equal to width and with relatively little difference between thickness of web and flanges, may not be representative of the many common sections whose depth is considerably larger than the width and/or whose flange thickness exceeds that of the web by a considerably larger percentage than for 8 WF 31. Cooling stresses are always so distributed that the portion which cools first shows residual compression stresses. It seems likely that in relatively deep and narrow sections, with relatively thick flanges, the web would cool first, resulting in a pattern of average residual stresses in web and flanges substantially the reverse from the authors'.

Before practical conclusions are drawn regarding the influence of residual cooling stresses on strength and performance of beam columns, a more complete investigation of the distribution of these stresses in the more common sections may be advisable.

Although neither the authors nor the writer have made any theoretical

4. C. P. Siess & I. M. Viest: Tests of Continuous Right I-Beam Bridges, U. of Ill. Exp. Sta. Bull. No. 416, 1953.
5. K. H. Rühl: Die Tragfähigekeit Metallischer Baukörper, W. Ernst & Sohn. Berlin, 1952, p. 66.

or experimental determination of the influence on column strength of a pattern of average residual stress substantially opposite to that of the paper, one may assume from general reasoning that such a stress pattern would not weaken and might even strengthen the member in eccentric compression. Since the evidence cited above makes it appear that in as-rolled sections patterns of either kind will probably be found, the explicit inclusion in design procedures of the influence of residual stress may turn out to be an extremely involved undertaking in view of the variety of probable stress patterns.

## Cornell Tests

Since the results of most of the authors' tests were influenced by premature twisting, they have attempted in Fig. 41 to correlate their theory with tests at Cornell University. These latter tests were carried out with bending by eccentricity produced about the minor axis for the precise purpose of eliminating twist. They represent part of an extensive analytical and experimental investigation by the writer and his colleagues which is described in Strength of Columns Elastically Restrained and Eccentrically Loaded, by P. P. Bijlaard, G. P. Fisher and George Winter, Proc. ASCE, Sep. No. 292, 1953. On request results of this investigation had been made available to the Lehigh investigators in advance of publication.

As indicated in the title, the Cornell investigation is concerned with columns elastically restrained at the ends, while the authors' theory is developed for pin-ended columns. This attempt to reconcile two different situations has caused some misunderstandings which make Fig. 41 invalid. The tests of this figure are those presented on Fig. 15a of the quoted Proc. Sep. No. 292.

In the theoretical part of that paper it is shown that elastically restrained, eccentrically loaded columns of slenderness ratio $L/r$ and eccentricity $e$ can be analyzed by replacing them by an equivalent hinged column of equivalent slenderness $cL/r$ and equivalent eccentricity $\mathcal{E}$. The methods for determining these two quantities are given in the paper.

For the restrained columns of Fig. 41 the actual slenderness ratio (between knife edges) was about $L/r = 80$, and the actual eccentricity ratios $e/r$ were as indicated in that figure. For the equivalent, hinged column, according to the theory of the Cornell paper, the effective slenderness was $cL/r = 50.2$, in close agreement with the value of 50 indicated by the authors. However, the equivalent eccentricity ratio according to that theory was $\mathcal{E}/r = 0.256(e/r)$. The authors did not take account of this fact in plotting $e/r$. Hence their figure refers in part to the equivalent hinged column ($cL/r$) and in part to the real, restrained column ($e/r$) and is, therefore, contradictory in itself. The apparently good agreement between test and theory shown on the figure is due to an accidental arithmetical oversight which happened to compensate almost exactly for the error introduced by using the wrong kind of eccentricity. This error was found independently by the authors and by the writer in his attempt to analyze Fig. 41.

It is hoped that in their closure the authors will present a revised evaluation of these tests, based on an application of their theory to the

concept developed in the Cornell paper of replacing a restrained column by an equivalent hinged column. If such a correct evaluation should show good correlation between theory and tests, it would represent a confirmation not only of the authors' analytical method for hinged columns, but also a further confirmation of the Cornell approach to the analysis of restrained, eccentric columns.

It may be worth mentioning that part of the Cornell tests were made on annealed specimens in order to eliminate residual stresses, while part were made on as-rolled specimens. The stress-strain curve for the unannealed I-sections on Fig. 13 of Proc. Sep. 292 if compared with that of the annealed I-sections, indicates by its rounded knee that residual stresses must have been present. The theory given in that paper does not explicitly account for the influence of residual stresses. It merely assumes that the compressive yield point as obtained by testing the entire section (rather than cut-out coupons) reflects in some measure the influence of these residual stresses. While this is known to be only approximate, the agreement between the Cornell theory and tests is equally good for the annealed and for the unannealed sections, at least in the tested range.

It should be added that the 4 I 9.5 section used in the Cornell tests is extremely stocky as compared to the authors' wide and thin-walled 8 WF 31. In view of what has been said above about residual stress patterns, it may be assumed that the residual stresses in the I-section were probably considerably smaller than in the WF-section, and also that their distribution pattern may have been quite different from that of the authors' section. The authors' statement on p. 330-12 relative to the possible influence of residual stress on the Cornell tests is, therefore, viewed with some reservation.

## SUMMARY

In summary the following may be said:

(a) The authors' analytical method seems to represent a valuable research tool for the investigation of beam columns without and with residual stresses.

(b) It would be desirable to have confirmation of this theory by tests specifically designed to exclude the influence of twist. This influence, which is not included in the authors' analysis, but which affected practically all their tests, makes it impossible to establish a firm correlation between theory and tests, within the scope of the paper.

(c) The experimental evidence, together with that of the quoted earlier Lehigh tests, indicates the sizeable weakening effect of twist even for relatively short columns bent about the strong axis. This influence seems to be of considerably greater importance than that of residual stress. The performance and particularly the strength of columns bent about the strong axis will be understood only when an adequate theory of flexural-torsional behavior in the plastic range will be available.

(d) There is some evidence that in rolled sections patterns of average residual stress in web and flanges may occur which are different

from and possibly even substantially opposite to the particular pattern measured by the authors for 8 WF 31 and assumed in their analysis. The details of the pattern seem to depend on the details of the cross-sectional dimensions. In view of this great variety of possible residual stress patterns, attempts to account for them rationally in design procedures may prove to be very complex.